Risolvere il seguente problema di Cauchy

$$\begin{cases} y' + \left(\dfrac{x+1}{x}\right) y = x & x > 0 \\ y(1) = 0 \end{cases}$$

Ricordando la formula risolutiva per le eq.ni differenziali lineari di 1° ordine :

$$y'(x) + p(x)y(x) = q(x) \quad \Rightarrow \quad y(x) = e^{-\int p(x)dx}\left[\int q(x) \cdot e^{-\int p(x)dx} dx + c\right]$$

si ha :

$$y(x) = e^{-\int \frac{x+1}{x}dx}\left[\int x \cdot e^{-\int \frac{x+1}{x}dx} dx + c\right] \quad \Rightarrow \quad y(x) = e^{-x-\ln x}\left[\int x \cdot e^{-x-\ln x} dx + c\right] \quad =$$

$$y(x) = e^{-x-\ln x}\left[\int e^{-x} dx + c\right] \quad =$$

$$y(x) = e^{-x-\ln x}\left[-e^{-x} + c\right]$$

Per Cauchy si ottiene :

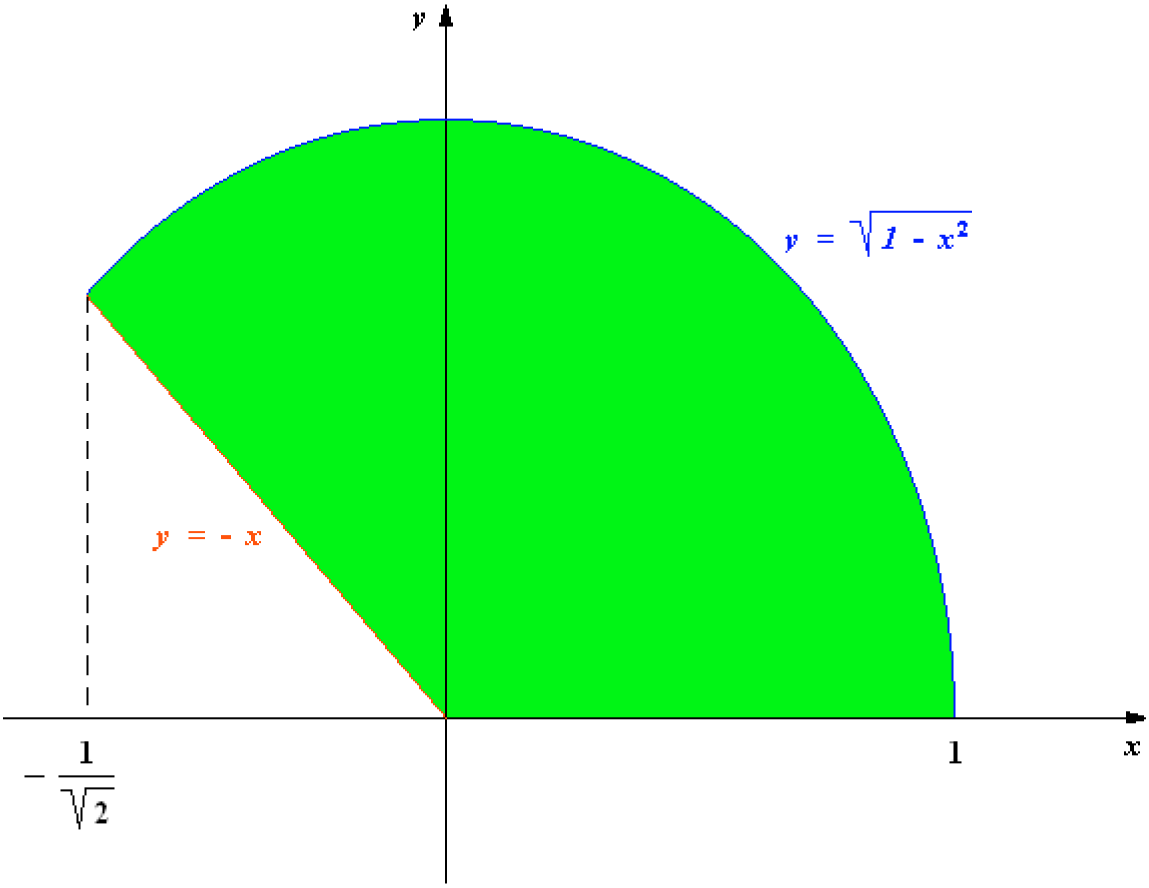
$$y(1) = 0 \quad \Rightarrow \quad 0 = e^{-1}\left[-e^{-1} + c\right] \quad \Rightarrow \quad c = e^{-1}$$

e quindi la soluzione di Cauchy : $y(x) = e^{-x-\ln x}\left[-e^{-x} - e^{-1}\right]$

Calcolare il seguente integrale doppio

$$\iint_D \frac{dxdy}{1+\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}+1\right)}$$

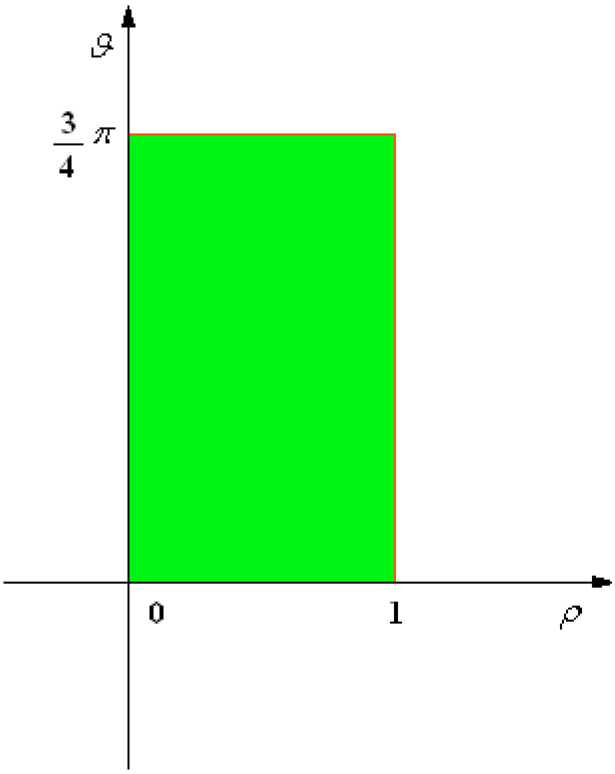
dove $D=\left\{(x,y)\in \Re^2 :\ -x\le y\le \sqrt{1-x^2}\ \right\}$



Utilizzando la trasformazione delle coordinate (cartesiane-polari) si ha :

$$\begin{cases} x=\rho\cos\vartheta \\ x=\rho\,\mathrm{sen}\,\vartheta \end{cases} \Rightarrow \quad 0\le\rho\le 1 \quad , \quad 0\le\vartheta\le\frac{3}{4}\pi$$

Il nuovo dominio T diventa quindi :



Considerando infine il dominio normale a ϑ si ha :

$$\iint_D \frac{dxdy}{1+\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}+1\right)} = \iint_T \frac{\rho d\rho d\vartheta}{1+\rho(\rho+1)}$$

$$\iint_T \frac{\rho d\rho d\vartheta}{1+\rho(\rho+1)} = \int_0^{\frac{3}{4}\pi} d\vartheta \int_0^1 \frac{\rho}{\rho^2+\rho+1} d\rho = \int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2} \int_0^1 \frac{2\rho+1-1}{\rho^2+\rho+1} d\rho \right)$$

$$\int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2} \int_0^1 \frac{2\rho+1}{\rho^2+\rho+1} d\rho - \frac{1}{2} \int_0^1 \frac{d\rho}{\rho^2+\rho+1} \right) = \int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2}\ln\left(\rho^2+\rho+1\right) - \frac{1}{2} \int_0^1 \frac{d\rho}{\left(\rho+\frac{1}{2}\right)^2+\frac{3}{4}} \right)_0^1$$

$$\int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2}\ln\left(\rho^2+\rho+1\right) - \frac{1}{2}\int_0^1 \frac{d\rho}{\left(\rho+\frac{1}{2}\right)^2+\frac{3}{4}} \right)_0^1 = \int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2}\ln\left(\rho^2+\rho+1\right) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\int_0^1 \frac{\frac{2}{\sqrt{3}}d\rho}{\left(\frac{2\rho+1}{\sqrt{3}}\right)^2+1} \right)_0^1 =$$

$$\int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2}\ln\left(\rho^2+\rho+1\right) - \frac{\sqrt{3}}{4}\operatorname{arctg}\left(\frac{2\rho+1}{\sqrt{3}}\right) \right)_0^1 = \int_0^{\frac{3}{4}\pi} d\vartheta \left(\frac{1}{2}\ln 3 - \frac{\sqrt{3}}{4}\cdot\frac{\pi}{3} + \frac{\sqrt{3}}{4}\cdot\frac{\pi}{6} \right) =$$

$$\left(\ln 9 - \frac{\sqrt{3}}{24}\pi \right) \int_0^{\frac{3}{4}\pi} d\vartheta = \frac{3}{4}\pi \left(\ln 9 - \frac{\sqrt{3}}{24}\pi \right)$$